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Application of supersymmetry to a coupled system of equations: the concept of a superpotential matrix

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Abstract. We apply supersymmetric quantum mechanics (SUSY QM) to multidimensional Schrödinger equations involving nonseparable potentials, which result in a system of coupled differential equations, where instead of the conventional definition of a scalar superpotential we introduce a superpotential matrix and succeed in applying SUSY QM to the coupled system. Finally, we discuss the shape-invariance condition for the potential matrix of such a system.

Supersymmetric quantum mechanics (SUSY QM) has provided an insight into the analytic solvability of the Schrödinger equation for certain classes of potentials. In this paper, we apply SUSY QM to the few-body Schrödinger equation with noncentral interactions, which results in a system of coupled differential equations.

It is well known that the Schrödinger equation for an arbitrary potential cannot be solved analytically. However, for a few well known potentials (such as the one-dimensional infinite square well, harmonic oscillator, Coulomb potential, etc) a complete analytic solution in a closed form can be obtained. In some cases (such as a finite square well, spherically symmetric three-dimensional infinite square well, etc) an analytic treatment is possible, but the energy eigenvalue is given by a transcendental equation and is not in a closed analytic form. Among the well known solvable potentials, the one-dimensional harmonic oscillator potential can easily be treated by introducing creation and destruction operators, which effectively factorizes the Hamiltonian. To gain insight into the analytical solvability, attempts were made to factorize the Hamiltonian for other potentials also. Schrödinger [1] introduced the factorization method to solve the hydrogen atom problem algebraically. This method was later generalized by Infeld and Hull [2] for a number of solvable potentials. In SUSY QM [3] the Hamiltonian is factorized by introducing a first-order differential operator (A) and its adjoint ($A^\dagger$) such that (after shifting the energy scale so that the ground state is at zero energy) $H = H_1 = A^\dagger A$. Then a SUSY partner Hamiltonian can be defined as $H_2 = AA^\dagger$. The partner Hamiltonian (H_2) has the same energy spectrum as the original (H_1), except that the lowest energy state of H_1 is missing in the spectrum of H_2. Infeld and Hull and later Gendenshtein [4] showed that if the partner potential (V_2) has the same shape as the original potential (V_1), then the energy spectrum of H_1 can be obtained algebraically in closed form. Such a potential is called a shape-invariant potential (SIP). It was later shown [5–8] that energy eigenfunctions and the scattering matrix can also be obtained algebraically for SIPs. For the most commonly known analytically solvable potentials listed in [9, 10], the shape invariance is obtained by a translation of parameters. Later the class of SIPs was broadened by including scaling of parameters [11].

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Most of the applications of SUSY QM and investigations of the shape invariance in SUSY QM have been done for the one-body Schrödinger equation for one-dimensional potentials and also for spherically symmetric potentials in three dimensions. In the latter case, the Schrödinger equation reduces to an ordinary differential equation in one variable (on the half-line). Some attempts have been made to include non-central potentials for one-body problems [12, 13]. However, the treatment includes only those non-central potentials for which the Schrödinger equation separates into three ordinary differential equation in (r, θ, ϕ), respectively. If the effective potentials in each of these variables, namely $V(r)$, $V(\theta)$ and $V(\phi)$ are SIPs, then each of the equations can be solved algebraically.

Attempts have also been made to apply SUSY QM to the three-body problem in one dimension [14] including a three-body force [15]. In one dimension, the relative motion of the three-body problem is described by two scalar variables, which can be mapped into polar coordinates (r, ϕ). The two- and three-body potentials [3, 16] were so chosen that the equations in terms of r and ϕ are separated and each one is an SIP.

The Schrödinger equation for a particle moving in a general noncentral potential which is not separable in any particular coordinate system can be reduced to a system of coupled differential equations (CDEs) in one radial variable (r) on the half-line, by expanding the wavefunction in the complete set of spherical harmonics. In a similar fashion, the few-body Schrödinger equation can also be reduced to a set of CDEs in the hyper-radial variable on the half-line, by expanding the wavefunction in the complete set of hyperspherical harmonics spanning the hyperangular space [17–19]. A finite set of CDEs is obtained by truncating the expansion basis, which is then solved by numerical techniques. No attempt has been made so far to formulate SUSY QM for such a system of CDE. The essential difficulty is in the definition of a superpotential. In the case of a system of N coupled differential equations, the effective potential becomes an $N \times N$ matrix, and the eigenfunction becomes an N-component column vector. In the case of an ordinary differential equation in one variable (say x), the superpotential $W(x)$ is defined as $-\psi_0'(x)/\psi_0(x)$ (in units such that $\hbar/\sqrt{2m} = 1$), where $\psi_0(x)$ is the ground state wavefunction. In the case of a system of N coupled differential equations, this definition is not possible, since ψ_0 and ψ_0' are column vectors and W must be a square matrix. In the following we introduce a modified definition of the superpotential matrix through a matrix equation, and show that the SUSY QM can be applied to the system of CDEs as well. Finally, we discuss the condition of shape invariance in this case.

The hyper-radial Schrödinger equation for a few-body system, after removal of the first derivative, has the general form

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - E\right]u_k(r) + \sum_{k'=1}^{N} V_{kk'}(r)\,u_{k'}(r) = 0 \qquad (k = 1, N) \tag{1}$$

where r is the hyper-radial variable, m is an effective mass and $V_{kk'}(r)$ is the coupling potential matrix. Choosing the units such that $\hbar/\sqrt{2m}$=1, the set of CDEs, equation (1), can be written in matrix form as

$$\left\{\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - E\right)[I] + [V(r)]\right\}|u(r)\rangle = 0 \tag{2}$$

where a symbol enclosed within a square bracket indicates an $N \times N$ square matrix and a symbol enclosed in a ket notation represents an N-component column vector. Similarly $\langle u(r)|$ represents a row vector, and $[\,I\,]$ is the unit matrix. We rename the potential matrix $[V_1]$ and denote the energy and column vector for the nth excited state by $E_n^{(1)}$ and $|u^{(n)}\rangle_1$, respectively. As is usual with SUSY QM, we choose the energy scale such that the ground state energy in

$[V_1]$ is zero ($E_0^{(1)} = 0$):

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}[I] + [V_1]\right)\left|u^{(0)}\right\rangle_1 = 0. \tag{3}$$

In SUSY QM for a one-dimensional ordinary differential equation, one defines a superpotential $W(x)$ as $-\psi_0'(x)/\psi_0(x)$, where $\psi_0(x)$ is the ground state wavefunction. As mentioned earlier, since $\left|u^{(0)}\right\rangle_1$ and $\left|u^{(0)\prime}\right\rangle_1$ are column vectors, this definition is not tenable for equation (3). Instead, we define a real-symmetric superpotential matrix $[W(r)]$ through the matrix equation

$$[W]\left|u^{(0)}\right\rangle_1 = -\left|u^{(0)\prime}\right\rangle_1 \tag{4}$$

where a prime denotes a differential with respect to the argument. Differentiating equation (4) with respect to r, we have

$$[W]\left|u^{(0)\prime}\right\rangle_1 + [W']\left|u^{(0)}\right\rangle_1 = -\left|u^{(0)\prime\prime}\right\rangle_1. \tag{5}$$

Equations (3)–(5), suggest the relation

$$[V_1] = \left[W^2\right] - [W']. \tag{6}$$

We next define the matrix operators:

$$[A] = \frac{\mathrm{d}}{\mathrm{d}r}[I] + [W] \qquad \left[A^\dagger\right] = -\frac{\mathrm{d}}{\mathrm{d}r}[I] + [W]. \tag{7}$$

Then for any column vector $|\psi\rangle$,

$$\begin{aligned}\left[A^\dagger\right][A]|\psi\rangle &= \left(-\frac{\mathrm{d}}{\mathrm{d}r}[I] + [W]\right)(|\psi'\rangle + [W]|\psi\rangle) \\ &= \left(\left[W^2\right] - [W']\right)|\psi\rangle - |\psi''\rangle \\ &= \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}[I] + [V_1]\right)|\psi\rangle \\ &\equiv [H_1]|\psi\rangle\end{aligned} \tag{8}$$

and we have

$$[H_1] = \left[A^\dagger\right][A]. \tag{9}$$

This shows that the Hamiltonian $[H_1]$ is factorizable.

Next, we define

$$\begin{aligned}[H_2] &= [A]\left[A^\dagger\right] \\ &\equiv -\frac{\mathrm{d}^2}{\mathrm{d}r^2}[I] + [V_2].\end{aligned} \tag{10}$$

Then using definition (7), we have

$$[V_2] = \left[W^2\right] + [W']. \tag{11}$$

From equations (7) and (4), we have

$$[A]\left|u^{(0)}\right\rangle_1 = \left|u^{(0)\prime}\right\rangle_1 + [W]\left|u^{(0)}\right\rangle_1 = 0. \tag{12}$$

Then, by equation (9)

$$[H_1]\left|u^{(0)}\right\rangle_1 = 0 \tag{13}$$

which is consistent with our choice of energy scale.

Next we consider the eigenvalue equation

$$[H_1]\left|u^{(n)}\right\rangle_1 = E_n^{(1)}\left|u^{(n)}\right\rangle_1 = \left[A^\dagger\right][A]\left|u^{(n)}\right\rangle_1. \tag{14}$$

Let $|u^{(n)}\rangle_2$ be the eigenstate of $[H_2]$, corresponding to the eigenvalue $E_n^{(2)}$:

$$[H_2]|u^{(n)}\rangle_2 = E_n^{(2)}|u^{(n)}\rangle_2 = [A][A^\dagger]|u^{(n)}\rangle_2. \tag{15}$$

Now,

$$[H_2][A]|u^{(n)}\rangle_1 = [A][A^\dagger][A]|u^{(n)}\rangle_1 = [A][H_1]|u^{(n)}\rangle_1 = E_n^{(1)}[A]|u^{(n)}\rangle_1. \tag{16}$$

Thus $[A]|u^{(n)}\rangle_1$ is an eigenfunction of $[H_2]$ corresponding to eigenvalue $E_n^{(1)}$.

Now $[A]|u^{(0)}\rangle_1$ vanishes (equation (12)) and being a trivial solution, cannot be the ground state of $[H_2]$. Since $E_n^{(1)}$ increase with increasing n, the ground state of $[H_2]$ will be proportional to $[A]|u^{(1)}\rangle_1$ corresponding to the eigenvalue $E_1^{(1)}$, which, by definition (15), is equal to $E_0^{(2)}$, being the ground state of $[H_2]$.

In general, we have

$$\begin{aligned} |u^{(n)}\rangle_2 &= C_n[A]|u^{(n+1)}\rangle_1 \\ E_n^{(2)} &= E_{n+1}^{(1)} \qquad (n = 0, 1, 2, \ldots) \end{aligned} \tag{17}$$

where C_n is a normalization constant. Assuming $|u^{(n)}\rangle_1$ to be normalized, we have

$$\begin{aligned} {}_2\langle u^{(n)}|u^{(n)}\rangle_2 &= |C_n|^2\ {}_1\langle u^{(n+1)}|[A^\dagger][A]|u^{(n+1)}\rangle_1 \\ &= |C_n|^2\ E_{n+1}^{(1)} \end{aligned} \tag{18}$$

by equation (9). Hence

$$C_n = \frac{1}{\sqrt{E_{n+1}^{(1)}}}. \tag{19}$$

Thus $[A]$ transforms an eigenket of $[H_1]$ into an eigenket of $[H_2]$, corresponding to one lower excitation. In a similar fashion, we can show that

$$|u^{(n+1)}\rangle_1 = \frac{1}{\sqrt{E_n^{(2)}}}[A^\dagger]|u^{(n)}\rangle_2. \tag{20}$$

Thus $[A^\dagger]$ transforms an eigenket of $[H_2]$ into an eigenket of $[H_1]$, corresponding to one higher excitation. Hence the energy spectrum of $[H_2]$ is identical to that of $[H_1]$, except that the ground level of $[H_1]$, namely $E_0^{(1)}$ does not appear in the energy spectrum of $[H_2]$.

So far, we have succeeded in formulating SUSY QM for a system of CDEs in analogy with the one-dimensional case. For a given Hamiltonian $[H_1]$ in the shifted energy scale (such that its ground state is at zero energy), one can define a partner Hamiltonian $[H_2]$, which has the same energy spectrum except for the absence of the ground state. Shifting the energy scale once again, such that the ground state of $[H_2]$ is at zero energy, one can repeat the process to define a third Hamiltonian $[H_3]$, which has the same energy spectrum as $[H_2]$, except for the absence of the ground state of $[H_2]$. Hence $[H_3]$ has the same energy spectrum as $[H_1]$, except for the absence of the first two states. This process can be repeated until all the states of $[H_1]$ are exhausted. Such a procedure is possible for any potential matrix $[V_1]$. Now if the partner potential $[V_2]$ has the same form as $[V_1]$, the potential $[V_1]$ will be called shape invariant. In such a case one can show that the energy spectrum and the eigenvectors can be obtained algebraically in closed forms.

The condition of shape invariance for a system of CDEs can be formulated more precisely as follows. A potential matrix $[V_1]$ is *shape invariant*, if it has the same functional form and matrix structure as that of the partner potential matrix $[V_2]$. Since $[V_1(r)]$ and $[V_2(r)]$ are

given by $\left[W^2(r)\right] - [W'(r)]$ and $\left[W^2(r)\right] + [W'(r)]$, respectively, one way of satisfying this condition will be if $[W(r)]$ has the form

$$[W(r)] = f(r)|a\rangle\langle a| \tag{21}$$

subject to the condition

$$\langle a|a\rangle = 1 \tag{22}$$

where $|a\rangle$ is an N-component column vector with constant elements $a_1, a_2, \ldots, a_N$. One immediately sees that $\left[W^2(r)\right]$ and $[W'(r)]$ and hence $[V_1(r)]$ and $[V_2(r)]$ have the same matrix structure, namely $|a\rangle\langle a|$. The potential matrices $[V_1(r)]$ and $[V_2(r)]$ are given by

$$\begin{aligned}[V_1(r)] &= \left(f^2(r) - f'(r)\right)|a\rangle\langle a| \equiv g_1(r)|a\rangle\langle a| \\ [V_2(r)] &= \left(f^2(r) + f'(r)\right)|a\rangle\langle a| \equiv g_2(r)|a\rangle\langle a|.\end{aligned} \tag{23}$$

If $g_1(r)$ and $g_2(r)$ satisfy the shape-invariance condition in one dimension [3], $[V_1(r)]$ and $[V_2(r)]$ are also shape-invariant potential matrices. If $g_1(r)$ is given in terms of a parameter c_1, then $g_2(r, c_1)$ can be obtained according to equation (23). Now if $g_2(r, c_1)$ satisfy the shape-invariance condition

$$g_2(r, c_1) = g_1(r, c_2) + R(c_1) \tag{24}$$

where c_2 is given in terms of c_1, then the shape-invariance condition in terms of the potential matrices is

$$[V_2(r, c_1)] = [V_1(r, c_2)] + R(c_1)|a\rangle\langle a|. \tag{25}$$

The Schrödinger equation satisfied by $[V_2(r, c_1)]$ is

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}[I] - E_n^{(2)}[I] + [V_2(r, c_1)]\right)\left|u^{(n)}\right\rangle_2 = 0. \tag{26}$$

Substituting from equations (23) and (24), equation (26) becomes

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}[I] - E_n^{(2)}[I] + g_1(r, c_2)|a\rangle\langle a| + R(c_1)|a\rangle\langle a|\right)\left|u^{(n)}\right\rangle_2 = 0. \tag{27}$$

Premultiplying with $\langle a|$ and using the condition (22), we obtain

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \left(E_n^{(2)} - R(c_1)\right) + g_1(r, c_2)\right)\left\langle a\middle|u^{(n)}\right\rangle_2 = 0 \tag{28}$$

which has the same form as the equation satisfied by $[V_1(r, c_1)]$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - E_n^{(1)} + g_1(r, c_1)\right)\left\langle a\middle|u^{(n)}\right\rangle_1 = 0. \tag{29}$$

Since both equations (28) and (29) correspond to the same effective potential $g_1(r)$, with different parameters, $E_n^{(1)}$ and $E_n^{(2)}$ are simply related. Then following the procedure for the one-dimensional shape invariance [3], one can calculate the energy spectrum $E_n^{(1)}$ algebraically. Starting from equation (28), one can again shift the energy scale so that the ground state of equation (28) is at zero energy. Then follow the procedure indicated by equations (23) and (24) to obtain $g_2(r, c_2)$ as $g_1(r, c_3) + R(c_2)$, where c_3 is the new parameter obtained as the same functional form of c_2 as c_2 was of c_1. Repeating in this manner, one can show [3] that

$$E_n^{(1)} = \sum_{k=1}^{n} R(c_k). \tag{30}$$

Thus we can obtain the excitation energies of the original Hamiltonian algebraically.

As a very simple example of this procedure, we take an N-component column vector for $|a\rangle$

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} \tag{31}$$

where $a_1, a_2, \ldots, a_N$ are constants, with a_i proportional to i.

Normalization gives

$$\langle a|a\rangle = \sum_{i=1}^{N} |a_i|^2 = 1. \tag{32}$$

Then the matrix part of $[W(x)]$ is

$$[M]_{ij} = a_i a_j = \frac{6}{N(N+1)(2N+1)} ij. \tag{33}$$

Furthermore, choosing $f(r)$ corresponding to the Eckart potential [3],

$$f(r) = -A \coth(br) + \frac{B}{A}. \tag{34}$$

With A as the first parameter (c_1), we have

$$g_1(r, c_1 = A) = A(A-b)\,\mathrm{cosech}^2(br) - 2B\coth(br) + \frac{B^2}{A^2} + A^2. \tag{35}$$

Taking $N = 3$, the three coupled differential equations have the matrix form

$$\begin{bmatrix} \left\{-\dfrac{\mathrm{d}^2}{\mathrm{d}r^2} - E_n^{(1)} + \frac{1}{14}g_1(r,c_1)\right\} & \frac{1}{7}g_1(r,c_1) & \frac{3}{14}g_1(r,c_1) \\ \frac{1}{7}g_1(r,c_1) & \left\{-\dfrac{\mathrm{d}^2}{\mathrm{d}r^2} - E_n^{(1)} + \frac{2}{7}g_1(r,c_1)\right\} & \frac{3}{7}g_1(r,c_1) \\ \frac{3}{14}g_1(r,c_1) & \frac{3}{7}g_1(r,c_1) & \left\{-\dfrac{\mathrm{d}^2}{\mathrm{d}r^2} - E_n^{(1)} + \frac{9}{14}g_1(r,c_1)\right\} \end{bmatrix} \times \begin{bmatrix} \left(u_1^{(n)}(r)\right)_1 \\ \left(u_1^{(n)}(r)\right)_2 \\ \left(u_1^{(n)}(r)\right)_3 \end{bmatrix} = 0 \tag{36}$$

where $\left(u_1^{(n)}(r)\right)_i$ is the ith component of the column vector $|u^{(n)}(r)\rangle_1$. By equation (23), $g_2(r, c_1)$ is obtained as

$$\begin{aligned} g_2(r, c_1 = A) &= A(A+b)\,\mathrm{cosech}^2(br) - 2B\coth(br) + \frac{B^2}{A^2} + A^2 \\ &= g_1(r, c_2 = A+b) + R(c_1 = A) \end{aligned} \tag{37}$$

where

$$R(c_1 = A) = \left\{\frac{B^2}{A^2} - \frac{B^2}{(A+b)^2}\right\} + \left\{A^2 - (A+b)^2\right\}. \tag{38}$$

Then repeating the procedure, we have

$$\begin{aligned} c_1 &= A \\ c_2 &= c_1 + b = A + b \\ c_3 &= c_2 + b = A + 2b \\ c_k &= A + (k-1)b \end{aligned} \tag{39}$$

$$R(c_k) = \left\{ \frac{B^2}{c_k^2} - \frac{B^2}{(c_k+b)^2} \right\} + \left\{ c_k^2 - (c_k+b)^2 \right\}. \tag{40}$$

Then the excited energy of equation (36) is given by equations (30) and (40),

$$E_n^{(1)} = \sum_{k=1}^{n} R(c_k) = \left\{ \frac{B^2}{A^2} - \frac{B^2}{(A+nb)^2} \right\} + \left\{ A^2 - (A+nb)^2 \right\}. \tag{41}$$

Next we see that the eigenvectors can be obtained analytically. Suppose that the nth excited state eigenfunction of equation (29) is $\psi_1^{(n)}(r)$:

$$\left\langle a \middle| u^{(n)}(r) \right\rangle_1 = \psi_1^{(n)}(r). \tag{42}$$

The one-dimensional supersymmetric quantum mechanical procedure gives the wavefunction $\psi_1^{(n)}(r)$ of equation (29) in an algebraic manner [3]. Then we can easily verify that

$$\left(u_1^{(n)}(r) \right)_i = a_i \psi_1^{(n)}(r) \qquad (i = 1, N) \tag{43}$$

satisfies equation (42). Hence all the components of $\left| u^{(n)} \right\rangle_1$ are obtained algebraically. Thus both the eigenvalue and the eigenket (N-component column vector) are obtained algebraically.

This simple example has been chosen to illustrate the procedure. As in the one-dimensional case, not all CDEs are algebraically solvable. The condition for solvability is the shape-invariance conditions expressed in the paragraph preceding equation (21). The form of $[W(r)]$ given by equation (21) is just *one* simple way of satisfying the shape-invariance condition.

We thus succeed in generalizing SUSY QM to a system of N coupled differential equations through the definition of a superpotential matrix. We have also shown that the shape-invariance condition for the system of CDEs can be formulated, which leads to an algebraic solution. A simple example illustrates the feasibility of satisfying this condition.

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